

J.D. Jackson Problem 9.16

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1 Power Radiated per Unit Solid Angle

We'll start by writing a functional form for the current density at all points in space. We know that the current in the rod is given by $I(z') = I_0 \sin kz' \hat{\mathbf{z}}$. But we have to dress this up as $\mathbf{J}(\mathbf{x})$. Note that the wave number $k = \frac{2\pi}{d}$.

$$\mathbf{J}(\mathbf{x}) = \delta(x)\delta(y)I(z') = I_0 \sin kz' [\Theta(z + d/2) - \Theta(z - d/2)] \hat{\mathbf{z}} \quad (1)$$

Using Equation 9.8 from Jackson, we can find the vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int_{AllSpace} \mathbf{J}(\mathbf{x}') e^{-ik\hat{\mathbf{n}} \cdot \mathbf{x}'} d^3\mathbf{x}' \quad (2)$$

In the geometry of our problem, $\hat{\mathbf{n}} \cdot \mathbf{x}' = z' \cos \theta$.

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} I_0 \sin kz' e^{-ikz' \cos \theta} \hat{\mathbf{z}} dz' \quad (3)$$

Now we'll use Euler's formula to expand the integral.

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} \sin(kz') \cos(kz' \cos \theta) - i \sin(kz') \sin(kz' \cos \theta) \hat{\mathbf{z}} dz' \quad (4)$$

The first term in the integral is zero because the sine function is odd and the cosine function is even.

$$\mathbf{A}(\mathbf{x}) = \frac{-i\mu_0 I_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} \sin(kz') \sin(kz' \cos \theta) \hat{\mathbf{z}} dz' \quad (5)$$

Using an obscure trig identity, $2 \sin s \sin t = \cos(s - t) - \cos(s + t)$, we can turn the product of sines into a difference of cosines which will be easier to integrate.

$$\mathbf{A}(\mathbf{x}) = \frac{-i\mu_0 I_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} \frac{1}{2} [\cos[kz'(1 - \cos \theta)] - \cos[kz'(1 + \cos \theta)]] dz' \quad (6)$$

Finally we can perform the actual integration.

$$\mathbf{A}(\mathbf{x}) = \frac{-i\mu_0 I_0}{8\pi k} \frac{e^{ikr}}{r} \hat{\mathbf{z}} \left[\frac{\sin[kz'(1 - \cos \theta)]}{1 - \cos \theta} - \frac{\sin[kz'(1 + \cos \theta)]}{1 + \cos \theta} \right]_{-d/2}^{d/2} \quad (7)$$

Simplifying, and substituting $k = 2\pi/d$ in the leading fraction,

$$\mathbf{A}(\mathbf{x}) = \frac{-i\mu_0 I_0 d}{8\pi^2} \frac{e^{ikr}}{r} \hat{\mathbf{z}} \left[\frac{\sin[\pi(1 - \cos \theta)]}{1 - \cos \theta} - \frac{\sin[\pi(1 + \cos \theta)]}{1 + \cos \theta} \right] \quad (8)$$

Simplifying the terms in square brackets is a bit long, and is covered separately in appendix A. If you're interested in a less rigorous solution, proceed happily to the final form below.

After all that simplification, we obtain this form for the vector potential.

$$\mathbf{A}(\mathbf{x}) = \frac{-i\mu_0 I_0 d}{4\pi^2} \frac{e^{ikr}}{r} \frac{\sin(\pi \cos \theta)}{\sin^2 \theta} \hat{\mathbf{z}} \quad (9)$$

Equation 9.16 in Jackson's text relates vector potential \mathbf{A} , and dipole moment, \mathbf{p} . We'll rearrange that equation to solve for \mathbf{p} .

$$\mathbf{p} = -\frac{r}{e^{ikr}} \frac{4\pi}{i\mu_0\omega} \mathbf{A}(\mathbf{x}) \quad (10)$$

Plugging in the vector potential that we found, we see that

$$\mathbf{p} = -\frac{r}{e^{ikr}} \frac{4\pi}{i\mu_0\omega} \left[\frac{-i\mu_0 I_0 d}{4\pi^2} \frac{e^{ikr}}{r} \frac{\sin(\pi \cos \theta)}{\sin^2 \theta} \right] \quad (11)$$

$$\mathbf{p} = \frac{I_0 d}{\pi\omega} \frac{\sin(\pi \cos \theta)}{\sin^2 \theta} \hat{\mathbf{z}} \quad (12)$$

Finally, to get the power distribution, we can use Equation 9.23 from the text.

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2 \theta \quad (13)$$

Plugging in our form for dipole moment,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \frac{\sin^2(\pi \cos \theta)}{\sin^4 \theta} \sin^2 \theta \quad (14)$$

Remembering $k = \frac{\omega}{c}$ and $d = \frac{2\pi}{k}$, and cancelling the appropriate terms,

$$\frac{dP}{d\Omega} = \frac{Z_0 I_0^2}{8\pi^2} \frac{\sin^2(\pi \cos \theta)}{\sin^2 \theta} \quad (15)$$

2 Total Power Radiated

The total power radiated is calculated by integrating over all solid angles.

$$P = \frac{Z_0 I_0^2}{8\pi^2} \int_0^{2\pi} \int_0^\pi \frac{\sin^2(\pi \cos \theta)}{\sin^2 \theta} \sin \theta \, d\theta \, d\phi \quad (16)$$

The azimuthal integral is easy because the radiation pattern is azimuthally symmetric.

$$P = \frac{Z_0 I_0^2}{4\pi} \int_0^\pi \frac{\sin^2(\pi \cos \theta)}{\sin^2 \theta} (\sin \theta \, d\theta) \quad (17)$$

Evaluating the altitudinal integral will require a substitution, $x = \cos \theta$ such that $dx = -\sin \theta \, d\theta$.

$$P = -\frac{Z_0 I_0^2}{4\pi} \int_1^{-1} \frac{\sin^2(\pi x)}{1-x^2} \, dx \quad (18)$$

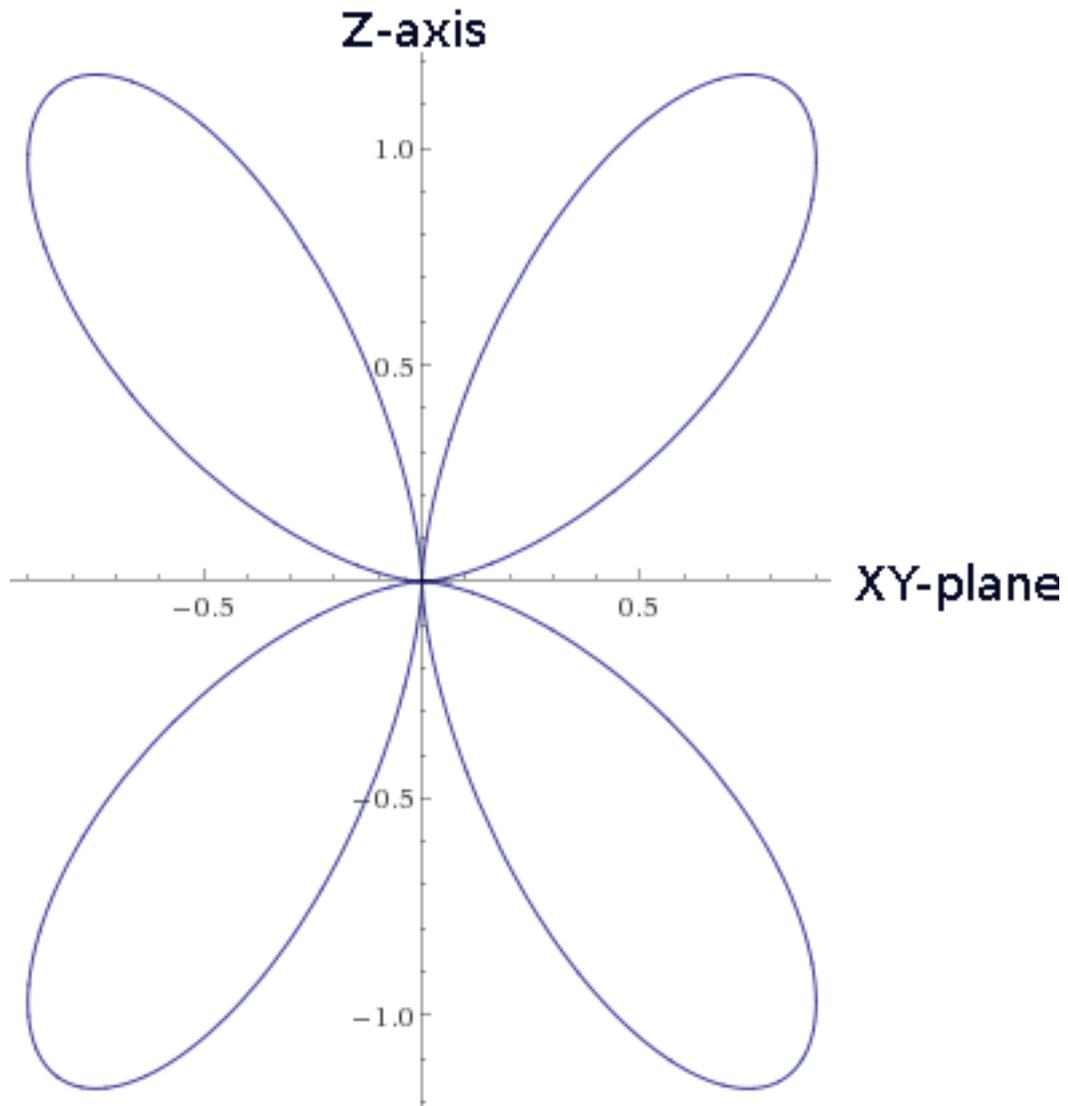
The closed form of the integral in question is not particularly insightful, but the result is.

$$P \approx \frac{Z_0 I_0^2}{4\pi} (1.57542) \quad (19)$$

To find the radiation resistance, we recall that $P = \frac{1}{2} I_0^2 R$ (is this in Jackson or something??).

$$R = \frac{2P}{I_0^2} = \frac{Z_0}{2\pi} (1.57542) = .2507 Z_0 \quad (20)$$

Figure 1: Altitudinal dependence of power in units of $\frac{Z_0 I_0^2}{8\pi}$.



A Simplifying the Vector Potential Integral

We'll start with this expression, and explain the simplifications line by line.

$$\frac{\sin[\pi z'(1 - \cos \theta)]}{1 - \cos \theta} - \frac{\sin[\pi z'(1 + \cos \theta)]}{1 + \cos \theta} \quad (21)$$

I'll first distribute the π inside the argument of the numerator.

$$\frac{\sin[\pi - \pi \cos \theta]}{1 - \cos \theta} - \frac{\sin[\pi + \pi \cos \theta]}{1 + \cos \theta} \quad (22)$$

The next step is to get a common denominator.

$$\frac{\sin(\pi(1 - \cos \theta))(1 + \cos \theta) - \sin(\pi(1 + \cos \theta))(1 - \cos \theta)}{1 - \cos^2 \theta} \quad (23)$$

Now we can use the angle difference formula, $\sin(s - t) = \sin s \cos t - \cos s \sin t$, to expand each term in the numerator. I'll also multiply the resulting binomials in this step.

$$\frac{1}{\sin^2 \theta} [\sin \pi \cos(\pi \cos \theta) - \cos \pi \sin(\pi \cos \theta) + \sin \pi \cos(\pi \cos \theta) \cos \theta - \cos \pi \sin(\pi \cos \theta) \cos \theta \\ - \sin \pi \cos(\pi \cos \theta) - \cos \pi \sin(\pi \cos \theta) + \sin \pi \cos(\pi \cos \theta) \cos \theta + \cos \pi \sin(\pi \cos \theta) \cos \theta] \quad (24)$$

The first, third, fifth, and seventh terms above contain $\sin \pi$ and are consequently zero. The remaining terms simplify slightly because they contain $\cos \pi = -1$.

$$\frac{1}{\sin^2 \theta} [\sin(\pi \cos \theta) + \sin(\pi \cos \theta) \cos \theta + \sin(\pi \cos \theta) - \sin(\pi \cos \theta) \cos \theta] \quad (25)$$

The second and fourth terms above cancel, while the first and third terms combine to give the final simplified form.

$$\frac{2}{\sin^2 \theta} \sin(\pi \cos \theta) \quad (26)$$

B The Two Wavelength Problem

Let's investigate how the problem would be different if two complete waves stood on the antenna instead of just one.