J.D. Jackson Problem 5.3

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We're asked to find the magnetic field due to a current density, so this is a typical application of the Biot-Savart law which is given as equation (5.4) in Jackson's text.

$$\mathrm{d}\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\mathbf{l}\times\mathbf{x}}{|\mathbf{x}|^3} \tag{1}$$

Using cylindrical coordinates with the point of interest at the origin, and z as the location of the contributing loop. We know that $\mathrm{d}l = a\mathrm{d}\theta$ and $|\mathbf{x}| = \sqrt{a^2+z^2}$, so the integral simplifies

$$\mathrm{d}\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{a\mathrm{d}\theta\sqrt{a^2+z^2}}{(a^2+z^2)^{3/2}} \tag{2}$$

We will first find the field due to a single loop. By symmetry, we know that the radial components of magnetic field will cancel, and only the z-component contributes, so $\mathrm{d}B_z = \sin\arctan\frac{a}{z}\mathrm{d}B = \frac{a}{\sqrt{a^2+z^2}}\mathrm{d}B$.

$$\mathrm{d}\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{a^2\mathrm{d}\theta}{(a^2+z^2)^{3/2}} \tag{3}$$

Integrating with respect to θ gives us the total magnetic field due to a single loop of current.

$$B_z = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{a^2\mathrm{d}\theta}{(a^2+z^2)^{3/2}} = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}} \tag{4}$$

Now we need to extend this result to account for contributions from all the loops instead of just one. We'll integrate over z. The number of turns in a small length is $N\mathrm{d}z$.

$$\mathrm{d}B_{z-tot} = \frac{\mu_0 NI}{2}\frac{a^2}{(a^2+z^2)^{3/2}}\mathrm{d}z \tag{5}$$

We have to write the bounds of integration in terms of the angles θ_1 and θ_2 as Jackson defines them in the problem. Note that the negative in front of the lower bound is due to the fact that Jackson defines θ_1 in a non-standard way.

$$B_{z-tot} = \frac{\mu_0 NIa^2}{2}\int_{-\frac{a}{\tan\theta_1}}^{\frac{a}{\tan\theta_2}}\frac{\mathrm{d}z}{(a^2+z^2)^{3/2}} \tag{6}$$

$$= \frac{\mu_0 NI}{2}\left[\frac{z}{\sqrt{a^2+z^2}}\right]_{-\frac{a}{\tan\theta_1}}^{\frac{a}{\tan\theta_2}} \tag{7}$$

$$= \frac{\mu_0 NI}{2}\left[\frac{1}{\sqrt{\tan^2\theta_2+1}} + \frac{1}{\sqrt{\tan^2\theta_1+1}}\right] \tag{8}$$

$$= \frac{\mu_0 NI}{2}\left[\frac{\cos\theta_2}{\sqrt{\sin^2\theta_2+\cos^2\theta_2}} + \frac{\cos\theta_2}{\sqrt{\sin^2\theta_1+\cos^2\theta_1}}\right] \tag{9}$$

$$\boxed{B_{z-tot} = \frac{\mu_0 NI}{2}[\cos\theta_2+\cos\theta_1]} \tag{10}$$

I really believe that this result is valid for any finite values of θ_1, θ_2, and N. I did not take any limits, although the result still holds in the limit as $NL\to\infty$.