J.D. Jackson Problem 5.13

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We'll set up our coordinate system so that the origin is at the center of the sphere, and the axis of rotation is along the positive z-axis. Now we can write the surface current density.

$$\mathbf{K} = \sigma \mathbf{v} = \sigma \omega \times \mathbf{r} = \sigma \omega r \sin\theta \hat{\phi} \tag{1}$$

Jackson shows us how to find the vector potential from a current density in equation (5.32), but he only lists the formula for 3D densities. The cooresponding 2D and 1D equations are just as valid (see David J Griffiths, "Introduction to Electrodynamics").

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}\,\mathrm{d}S}{|\mathbf{r} - \mathbf{r}'|} \tag{2}$$

Substituting our form of K and using the right hand rule to determine direction,

$$\mathbf{A} = \frac{\mu_0 \sigma \omega}{4\pi} \iint \frac{a^3 \sin^2\theta'}{|\mathbf{r} - \mathbf{r}'|} \,\mathrm{d}\theta' \,\mathrm{d}\phi' \,\hat{\phi}' \tag{3}$$

We can now use the expansion in spherical harmonics, equation (3.70), to replace that denominator

$$\mathbf{A} = \mu_0 \sigma \omega \sum_{l,m} \frac{a^3 r_<^l Y_{lm}(\theta, \phi)}{r_>^{l+1}(2l+1)} \iint \sin^2\theta' \, Y_{lm}^*(\theta', \phi') \,\mathrm{d}\theta' \,\mathrm{d}\phi' \,\hat{\phi}' \tag{4}$$

The problem with the vector direction $\hat{\phi}'$ is that it is not a fixed direction, it depends on the vector value of r. In order to resolve this, we must recast it into the fixed directions $\hat{\phi}' = \cos\phi' \hat{x} + \sin\phi' \hat{y}$.

$$\mathbf{A} = \mu_0 \sigma \omega \sum_{l,m} \frac{a^3 r_<^l Y_{lm}(\theta, \phi)}{r_>^{l+1}(2l+1)} \iint \sin^2\theta' \, Y_{lm}^*(\theta', \phi') [\cos\phi' \hat{x} + \sin\phi' \hat{y}] \,\mathrm{d}\theta' \,\mathrm{d}\phi' \tag{5}$$

Now we note the following useful identities

$$\sin\theta' \sin\phi' = -\sqrt{\frac{8\pi}{3}} Im(Y_{11}(\theta', \phi')) \tag{6}$$

$$\sin\theta' \cos\phi' = -\sqrt{\frac{8\pi}{3}} Re(Y_{11}(\theta', \phi')) \tag{7}$$

For brevity here, I'm only going to evaluate the $\hat{y}$ component because both integrals will be nearly identical. We'll remember to throw that other one back in at the end.

$$A_y = -\sqrt{\frac{8\pi}{3}}\mu_0\sigma\omega\sum_{l,m}\frac{a^3 r_<^l}{r_>^{l+1}(2l+1)} Im\left[Y_{lm}(\theta,\phi)\iint Y_{11}(\theta',\phi')\,Y_{lm}^*(\theta',\phi')\sin\phi'\,\mathrm{d}\theta'\,\mathrm{d}\phi'\right] \tag{8}$$

$$A_y = -\sqrt{\frac{8\pi}{3}}\mu_0\sigma\omega\sum_{l,m}\frac{a^3 r_<^l}{r_>^{l+1}(2l+1)} Im\left[Y_{lm}(\theta,\phi)\delta_{l1}\delta_{m1}\right] \tag{9}$$

$$A_y = -\sqrt{\frac{8\pi}{3}}\mu_0\sigma\omega\frac{a^3 r_<}{3r_>^2}\left[-\sqrt{\frac{3}{8\pi}}\sin\theta\sin\phi\right] \tag{10}$$

$$A_y = \mu_0\sigma\omega\frac{a^3 r_<}{3r_>^2}\sin\theta\sin\phi \tag{11}$$

$$\mathbf{A} = \mu_0\sigma\omega\frac{a^3 r_<}{3r_>^2}\sin\theta\left[\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right] \tag{12}$$

$$\mathbf{A} = \mu_0\sigma\omega\frac{a^3 r_<}{3r_>^2}\sin\theta\,\hat{\phi} \tag{13}$$

$$\tag{14}$$

Now we can just substitute the appropriate values of $r_<$ and $r_>$ for inside and outside.

$$\mathbf{A}_{inside} = \frac{1}{3}\mu_0\sigma\omega r a\sin\theta\,\hat{\phi} \tag{15}$$

$$\mathbf{A}_{outside} = \frac{1}{3}\mu_0\sigma\omega\frac{a^4}{r^2}\sin\theta\,\hat{\phi} \tag{16}$$

Finally to calculate the B-field we can use $\mathbf{B} = \nabla\times\mathbf{A}$.

$$\mathbf{B}_{inside} = \frac{2}{3}\mu_0\sigma\omega a[\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\theta}] \tag{17}$$

$$\mathbf{B}_{outside} = \frac{a^4}{3r^3}\mu_0\sigma\omega[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}] \tag{18}$$