J.D. Jackson Problem 4.6

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1 Energy of quadrupole interaction

The correct solution of this problem relies heavily on little factoids that Jackson drops on us with little justification. First he tells us that ρ is cylindrically symetric. Second he provides an external field that is cylindrically symetric about the same axis. Third he define $Q \equiv \frac{1}{e} Q_{33}$. So for this part there isn't a whole lot left to do.

We start with the quadripole term of the work equation (4.24) in the text.

$$W = -\frac{1}{6} \sum_{1} \sum_{j} Q_{ij} \frac{\partial E_j}{\partial x_i}(0) \tag{1}$$

Another little nugget of truth that Jackson gives us is that all the Q_{ij}'s are zero except for Q_{33}. But, importantly, he footnotes on page 151 that really all the diagonal terms are nonzero such that $Q_{11} = Q_{22} = -\frac{1}{2} Q_{33}$. We can also use Gauss's law for electrostatics to show that,

$$\nabla \cdot \mathbf{E} = \mathbf{0} \tag{2}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{3}$$

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = -\frac{1}{2} \frac{\partial E_z}{\partial z} \tag{4}$$

So the work equation can be written more simply.

$$W = \frac{1}{6} \left[\frac{-1}{2} Q_{33} \frac{-1}{2} \frac{\partial E_z}{\partial z} + \frac{-1}{2} Q_{33} \frac{-1}{2} \frac{\partial E_z}{\partial z} + Q_{33} \frac{\partial E_z}{\partial z} \right] \tag{5}$$

$$W = \frac{-1}{6} Q_{33} \frac{\partial E_z}{\partial z} \left[\frac{1}{4} + \frac{1}{4} + 1 \right] \tag{6}$$

$$\frac{-1}{4} Q_{33} \frac{\partial E_z}{\partial z} \tag{7}$$

Substituting the definition of quadripole moment,

$$\frac{-e}{4} Q \frac{\partial E_z}{\partial z} \tag{8}$$

2 An example with numbers

I didn't do this part. Maybe I'll get back to it eventually.

3 Elipsoidal nucleus model

Starting with equation (4.9) or (4.25),

$$Q = \frac{1}{e}\int 3z^2 - \left(\sqrt{x^2+y^2+z^2}\right)^2 \rho(\mathbf{x})\,\mathrm{d}V \tag{9}$$

We know that ρ is constant from the problem, so its value is $\rho = \frac{Ze}{V}$ Where the volume of the elipsoid is $V = \frac{4}{3}\pi ab^2$. We should note here that I have chosen a single major axis oriented along the z-axis. It would have been equally valid to choose a single minor axis along the z-axis and two major axes in the xy-plane.

$$Q = \frac{3Z}{4\pi ab^2}\int 2z^2 - x^2 - y^2\,\mathrm{d}V = \frac{3Z}{4\pi ab^2}\int 2z^2 - r^2\,\mathrm{d}V \tag{10}$$

See the final section for how to get the bounds of integration.

$$Q = \frac{3Z}{4\pi ab^2}\int_{-a}^{a}\int_{0}^{b\sqrt{1-\frac{z^2}{a^2}}}\int_{0}^{2\pi} 2z^2r - r^3\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z \tag{11}$$

Performing the first two integrations,

$$Q = \frac{3Z}{2ab^2}\int_{-a}^{a} z^2r^2 - \frac{1}{4}r^4\bigg|_0^{b\sqrt{1-\frac{z^2}{a^2}}}\,\mathrm{d}z \tag{12}$$

$$Q = \frac{3Z}{2ab^2}\int_{-a}^{a} z^2b^2\left(1-\frac{z^2}{a^2}\right) - \frac{1}{4}b^4\left(1-\frac{z^2}{a^2}\right)^2\,\mathrm{d}z \tag{13}$$

$$Q = \frac{3z}{2a}\int_{-a}^{a} z^2 - \frac{z^4}{a^2} - \frac{b^2}{4} + \frac{b^2z^2}{2a^2} - \frac{b^2z^4}{4a^4}\,\mathrm{d}z \tag{14}$$

$$Q = \frac{3z}{a}\left[\frac{-b^2a}{4} + \frac{a^3}{3} + \frac{b^2a}{6} - \frac{a^3}{5} - \frac{b^2a}{20}\right] \tag{15}$$

$$Q = \frac{Z}{20}(-8b^2 + 8a^2) \tag{16}$$

$$\boxed{Q = \frac{2Z}{5}(a^2 - b^2)} \tag{17}$$

Next Jackson wants us to solve for the factor $\frac{a-b}{R}$.

$$Q = \frac{4Z}{5}(a-b)\frac{a+b}{2} = \frac{4Z}{5}(a-b)R = \frac{4Z}{5}\frac{a-b}{R}R^2 \tag{18}$$

Solving for the desired factor,

$$\frac{a-b}{R} = \frac{5Q}{4ZR^2} \tag{19}$$

And plugging in the given values,

$$\frac{a-b}{R} = \frac{5\cdot(2.5E^{-28})}{4\cdot 63\cdot(7.0E^{-15})^2} \approx 0.10123 \tag{20}$$

A Bounds of integration

It's well established (for example check out wikipedia) that the equation for an elipsoid aligned along the z-axis is

$$\frac{x^2 + y^2}{b^2} + \frac{z^2}{a^2} = 1 \tag{21}$$

We want to integrate over the volume of the elipsoid, so the *theta* inegral is easy enough. We just go all the way around the xy-plane. I'll choose to make the z integral easy too by integrating from $-a$ to a. This means that all we need to do is solve the elipsoid equation for r to get the bounds for z.

$$\frac{r^2}{b^2} = 1 - \frac{z^2}{a^2} \tag{22}$$

$$r = b\sqrt{1 - \frac{z^2}{a^2}} \tag{23}$$

So we will integrate the r coordinate from 0 to the critical r value we just found.