J.D. Jackson Problem 4.1

Josh Orndorff

admin@joshorndorff.com

April 13, 2012

1   Charges in the xy–plane

The first step here is to write the charge distribution ρ in spherical coordinates. I'll first write ρ_1, the charge distribution of the single point charge on the positive x-axis. We know that the distribution will contain several delta functions and be proportional to q.

$$\rho_1 = Aq\delta(r - a)\delta(\theta - \frac{\pi}{2})\delta(\phi) \tag{1}$$

To find the proportionality constant, A, we will integrate both sides over all space knowing that the total integral is equal to the charge, q.

$$q = \int_0^{2\pi} \int_0^\pi \int_0^\infty Aq\delta(r - a)\delta(\theta - \frac{\pi}{2})\delta(\phi)r^2 \sin\theta \, dr \, d\theta \, d\phi \tag{2}$$

$$1 = \int_0^\pi A\delta(\theta - \frac{\pi}{2}) \sin\theta \, d\theta \tag{3}$$

$$A = \frac{1}{2\pi a^2} \tag{4}$$

Plugging this back in, we get the final form of this charge distribution.

$$\rho_1 = \frac{q}{2\pi a^2}\delta(r - a)\delta(\theta - \frac{\pi}{2})\delta(\phi) \tag{5}$$

Note that it would be just as acceptable, and more general, to use $A = \frac{1}{2\pi r^2}$ as the delta functions would pick out the correct values of r upon integration. Using the more general form which includes r rather than a will also solve the problem if a charge is placed at the origin (i.e, $r = 0$). I mention this here because we will rely on this logic to handle the θ dependence in the next section. We can now proceed to write the total charge distribution due to all four point charges.

$$\rho = \frac{q}{2\pi a^2}\delta(r - a)\delta(\theta - \frac{\pi}{2}) \left[\delta(\phi) + \delta(\phi - \frac{\pi}{2}) - \delta(\phi - \pi) - \delta(\phi - \frac{3\pi}{2}) \right] \tag{6}$$

We can calculate the q_{lm}'s by applying equation 4.3 from Jackson to calculate, $q_{lm} = \int Y_{lm}^* r^l \rho dV$.

$$q_{lm} = \frac{q}{2\pi a^2} \int_0^{2\pi} \int_0^\pi \int_0^\infty Y_{lm}^*(\theta, \phi)r^{l+2}\delta(r-a)\delta(\theta-\frac{\pi}{2}) \left[\delta(\phi) + \delta(\phi - \frac{\pi}{2}) - \delta(\phi - \pi) - \delta(\phi - \frac{3\pi}{2}) \right] \sin\theta dr d\theta d\phi \tag{7}$$

After evaluating the r integral,

$$q_{lm} = \frac{qa^l}{2\pi} \int_0^{2\pi} \int_0^\pi Y_{lm}^*(\theta, \phi)\delta(\theta - \frac{\pi}{2}) \left[\delta(\phi) + \delta(\phi - \frac{\pi}{2}) - \delta(\phi - \pi) - \delta(\phi - \frac{3\pi}{2}) \right] \sin\theta d\theta d\phi \tag{8}$$

After evaluating the θ integral,

$$q_{lm} = \frac{qa^l}{2\pi} \int_0^{2\pi} Y_{lm}^*(\frac{\pi}{2}, \phi) \left[\delta(\phi) + \delta(\phi - \frac{\pi}{2}) - \delta(\phi - \pi) - \delta(\phi - \frac{3\pi}{2}) \right] d\phi \tag{9}$$

After evaluating the ϕ integral,

$$q_{lm} = \frac{qa^l}{2\pi}\left[Y_{lm}^*(\frac{\pi}{2},0) + Y_{lm}^*(\frac{\pi}{2},\frac{\pi}{2}) - Y_{lm}^*(\frac{\pi}{2},\pi) - Y_{lm}^*(\frac{\pi}{2},\frac{3\pi}{2})\right] \tag{10}$$

Technically that does it, but we can go a little further and simplify those Y_{lm}^*'s using equation 3.53 in Jackson. After evaluating the ϕ integral,

$$q_{lm} = \frac{qa^l}{2\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\frac{\pi}{2})\left[e^{-im0} + e^{-im\frac{\pi}{2}} - e^{-im\pi} - e^{-im\frac{3\pi}{2}}\right] \tag{11}$$

$$q_{lm} = \frac{qa^l}{2\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1 + \left(e^{-i\frac{\pi}{2}}\right)^m - \left(e^{-i\pi}\right)^m - \left(e^{-i\frac{3\pi}{2}}\right)^m\right] \tag{12}$$

$$q_{lm} = \frac{qa^l}{2\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1 + (-i)^m - (-1)^m - (i)^m\right] \tag{13}$$

Considering only the portion in brackets, we see that the q_{lm}'s are non-zero only when m is odd, and that allows further simplification.

$$\boxed{q_{lm} = \frac{qa^l}{\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(1-i^m)P_l^m(0)} \tag{14}$$

The first few q_{lm}'s are,

$$q_{1,1} = \frac{-qa}{2\pi}\sqrt{\frac{3}{2\pi}}(1-i) \tag{15a}$$

$$q_{1,-1} = \frac{qa}{2\pi}\sqrt{\frac{3}{2\pi}}(1+i) \tag{15b}$$

2 Charges on the z–axis

As mentioned above, the charge distribution can be written with an r^2 and by the same logic $\sin\theta$ in the denominator. This troubled me at first, but integrating such a charge distribution over *any* region results in the correct total charge.

$$\rho = \frac{q}{2\pi\sin\theta r^2}\left[\delta(r-a)\left(\delta(\theta) + \delta(\theta-\pi)\right) - 2\delta(r)\right] \tag{16}$$

$$q_{lm} = \frac{q}{2\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}Y_{lm}^*(\theta,\phi)r^l\left[\delta(r-a)\left(\delta(\theta) + \delta(\theta-\pi)\right) - 2\delta(r)\right]\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \tag{17}$$

In this case, we know that $m = 0$ because of azimuthal symmetry so the spherical harmonics simplify.

$$q_{l0} = \frac{q}{2\pi}\sqrt{\frac{2l+1}{4\pi}}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}P_l(\cos\theta)r^l\left[\delta(r-a)\left(\delta(\theta) + \delta(\theta-\pi)\right) - 2\delta(r)\right]\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \tag{18}$$

Evaluating the integrals is relatively straight-forward, except for that last delta in the case of $l = 0$. In this case, $r^l = 0^0$ which is undefined in general. However, Jackson lists the proper value to use as equation 4.4.

$$q_{l0} = qa^l\sqrt{\frac{2l+1}{4\pi}}\left[P_l(\cos 0) + P_l(\cos\pi)\right] - \frac{2q}{\sqrt{4\pi}}\delta_{l,0} \tag{19}$$

$$q_{l0} = qa^l\sqrt{\frac{2l+1}{4\pi}}\left[1 + (-1)^l\right] - \frac{2q}{\sqrt{4\pi}}\delta_{l,0} \tag{20}$$

As before, examining the bracketed part of the equation shows that only even l's result in non-zero $q_l 0$'s. When l is even,

$$q_{l0} = 2qa^l \sqrt{\frac{2l+1}{4\pi}} - \frac{2q}{\sqrt{4\pi}} \delta_{l,0} \tag{21}$$

$$\boxed{q_{l0} = \frac{2q}{\sqrt{4\pi}} (a^l \sqrt{2l+1} - \delta_{l,0})} \tag{22}$$

The first few non-zero $q_l 0$'s are,

$$q_{2,0} = qa^2 \sqrt{\frac{5}{\pi}} \tag{23a}$$

$$q_{4,0} = qa^4 \sqrt{\frac{9}{\pi}} \tag{23b}$$

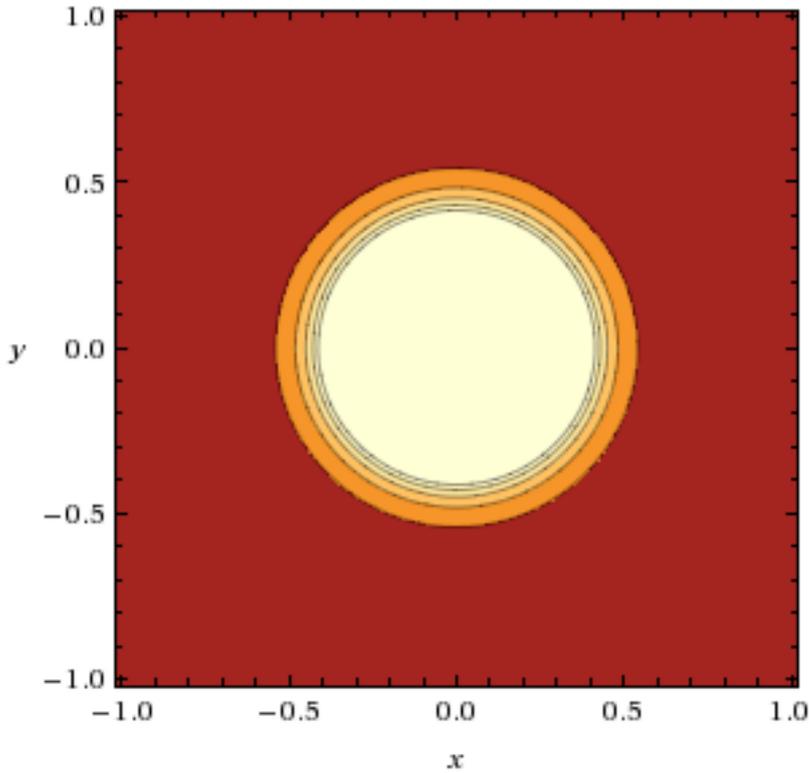# 3 Multipole expansion for electric potential

The proper expansion is given as equation 4.1 in the text. Substituting the form for $q_l 0$ that we just found and simplifying the Y_{lm} for $m = 0$ gives,

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_l \frac{4\pi}{2l+1} \left[\frac{2q}{\sqrt{4\pi}} (a^l \sqrt{2l+1} - \delta_{l,0}) \right] \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \tag{24}$$

$$\Phi = \frac{2q}{4\pi\epsilon_0} \sum_l (a^l - \delta_{l,0}) P_l(\cos\theta) \tag{25}$$

The lowest non-zero term is the $l = 2$ term, so in the xy–plane,

$$\Phi_2(r) = \frac{qa^2}{4\pi\epsilon_0 r^3} \qquad \Phi_2(a) = \frac{q}{4\pi\epsilon_0 a} \qquad \lim_{r\to\infty} \Phi_2(r) = 0 \tag{26}$$

4 Exact calculation of electric potential

Calculating the exact potential from Coulomb's law for this potential is actually quite a bit easier. I'll do the calculation for the negative charge and then for one of the positive charges.

$$\Phi_- = \frac{-2q}{4\pi\epsilon_0 r} \tag{27}$$

$$\Phi_+ = \frac{q}{4\pi\epsilon_0 \sqrt{a^2 + r^2}} \tag{28}$$

$$\Phi_{total} = \Phi_- + 2\Phi_+ = \frac{q}{2\pi\epsilon_0}\left(\frac{1}{\sqrt{a^2 + r^2}} - \frac{1}{r}\right) \tag{29}$$