

## J.D. Jackson Problem 3.9

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I'll begin this problem by assuming that  $\Phi$  is of a separable form,  $\Phi(r, \theta, z) = R(r)\Theta(\theta)Z(z)$  in cylindrical coordinates. Applying the Laplacian operator gives three ordinary differential equations subject to five constraints.

Five boundary constraints for separated solutions

$$\Phi = 0 @ z = 0 \tag{1a}$$

$$\Phi = 0 @ z = L \tag{1b}$$

$$\Phi \neq \pm\infty @ r = 0 \tag{1c}$$

$$\Phi = V(\theta, z) @ r = b \tag{1d}$$

$$\Phi(r, 0, z) = \Phi(r, 2\pi, z) \tag{1e}$$

Solving each of the three ordinary differential equations and naming the constants in the style of Jackson section 3.7 we have

$$\frac{Z''(z)}{Z(z)} = -k^2 \Rightarrow Z(z) = A_z \sin kz + B_z \cos kz \tag{2}$$

Constraints 1 and 2 show that  $B_z = 0$  and  $k = \frac{n\pi}{L}$  where  $n$  is a positive integer.

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -\nu^2 \Rightarrow \Theta(\theta) = A_\theta \sin \nu\theta + B_\theta \cos \nu\theta \tag{3}$$

Constraint 5 shows that  $\nu$  is a non-negative integer.

$$R'' + \frac{R'}{r} - \left(k^2 + \frac{\nu^2}{r^2}\right) R = 0 \tag{4}$$

Here the primes denote derivative with respect to  $r$ . Now, we'll follow Jackson and make the substitution  $x = kr$ ,  $dx = kdr$

$$\ddot{R} + \frac{\dot{R}}{x} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0 \Rightarrow R(x) = A_r I_\nu(x) + B_r K_\nu(x) \tag{5}$$

Constraint 3 shows that  $B_r = 0$ .

So putting the whole solution back together

$$\Phi = \sum_{n,\nu} I_\nu \left(\frac{n\pi r}{L}\right) \sin \left(\frac{n\pi z}{L}\right) [A_{n\nu} \sin(\nu\theta) + B_{n\nu} \cos(\nu\theta)] \tag{6}$$

All that is left to do now is calculate the  $A_{n\nu}$ 's and  $B_{n\nu}$ 's which can be accomplished by applying the final boundary condition.

$$V(\theta, z) = \sum_{n,\nu} I_\nu \left(\frac{n\pi b}{L}\right) \sin \left(\frac{n\pi z}{L}\right) [A_{n\nu} \sin(\nu\theta) + B_{n\nu} \cos(\nu\theta)] \tag{7}$$

Multiply both sides by  $\cos(\nu'\theta)$  and integrate

$$\int_0^{2\pi} V(\theta, z) \cos(\nu'\theta) d\theta = \sum_{n,\nu} I_\nu \left(\frac{n\pi b}{L}\right) \sin \left(\frac{n\pi z}{L}\right) \int_0^{2\pi} A_{n\nu} \sin(\nu\theta) \cos(\nu'\theta) + B_{n\nu} \cos(\nu\theta) \cos(\nu'\theta) d\theta \tag{8}$$

The  $A_{n\nu}$  terms are all zero because the sine and cosine product integrates to zero for any values of  $\nu$  and  $\nu'$ . The  $B_{n\nu}$  terms integrate to zero in every case except when  $\nu = \nu'$ .

$$\int_0^{2\pi} V(\theta, z) \cos(\nu'\theta) d\theta = \sum_{n,\nu} I_\nu \left( \frac{n\pi b}{L} \right) \sin \left( \frac{n\pi z}{L} \right) \pi B_{n\nu} \delta_{\nu\nu'} \quad (9)$$

The kronecker delta collapses the sum over  $\nu$

$$\int_0^{2\pi} V(\theta, z) \cos(\nu'\theta) d\theta = \pi \sum_n B_{n\nu'} I_{\nu'} \left( \frac{n\pi b}{L} \right) \sin \left( \frac{n\pi z}{L} \right) \quad (10)$$

Multiply both sides by  $\sin\left(\frac{n'\pi z}{L}\right)$  and integrate

$$\int_0^L \int_0^{2\pi} V(\theta, z) \cos(\nu'\theta) \sin \left( \frac{n'\pi z}{L} \right) d\theta dz = \pi \sum_n B_{n\nu'} I_{\nu'} \left( \frac{n\pi b}{L} \right) \int_0^L \sin \left( \frac{n\pi z}{L} \right) \sin \left( \frac{n'\pi z}{L} \right) dz \quad (11)$$

As with the cosines above, the sines are orthogonal and the integral is only non-zero when  $n = n'$ .

$$\int_0^L \int_0^{2\pi} V(\theta, z) \cos(\nu'\theta) \sin \left( \frac{n'\pi z}{L} \right) d\theta dz = \pi \sum_n B_{n\nu'} I_{\nu'} \left( \frac{n\pi b}{L} \right) \frac{L}{2} \delta_{nn'} \quad (12)$$

The kronecker delta collapses the sum over  $n$

$$\int_0^L \int_0^{2\pi} V(\theta, z) \cos(\nu'\theta) \sin \left( \frac{n'\pi z}{L} \right) d\theta dz = \frac{\pi L}{2} B_{n'\nu'} I_{\nu'} \left( \frac{n'\pi b}{L} \right) \quad (13)$$

Finally, solving for  $B_{n\nu}$  and noting that the  $A_{n\nu}$ 's can be calculated in the exact same way, we have,

$$A_{n\nu} = \frac{2}{L\pi I_\nu \left( \frac{n\pi b}{L} \right)} \int_0^{2\pi} \int_0^L \sin \left( \frac{n\pi z}{L} \right) \sin(\nu\theta) V(\theta, z) dz d\theta \quad (14a)$$

$$B_{n\nu} = \frac{2}{L\pi I_\nu \left( \frac{n\pi b}{L} \right)} \int_0^{2\pi} \int_0^L \sin \left( \frac{n\pi z}{L} \right) \cos(\nu\theta) V(\theta, z) dz d\theta \quad (14b)$$