J.D. Jackson Problem 3.18

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I'll begin this problem by assuming that Φ is of a seperable form, $\Phi(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ in cylinderical coordinates. Applying the Laplacian operator gives three ordinary differential equations subject to five constraints.

Five boundary constraints for separated solutions

$$\Phi = 0 @ z = 0 \tag{1a}$$
$$\Phi = V(r) @ z = L \tag{1b}$$
$$\Phi \neq \pm\infty @ r = 0 \tag{1c}$$
$$\Phi \to 0 @ r \to \infty \tag{1d}$$
$$\Phi(r,0,z) = \Phi(r,2\pi,z) \tag{1e}$$

Solving each of the three ordinary diferential equations and naming the constants in the style of Jackson section 3.7 we have

$$\frac{Z''(z)}{Z(z)} = k^2 \Rightarrow Z(z) = A_z \sinh kz + B_z \cosh kz \tag{2}$$

Constraint 1 shows that $B_z = 0$.

$$\frac{\Theta''\theta)}{\Theta(\theta)} = -\nu^2 \Rightarrow \Theta(\theta) = A_\theta \sin\nu\theta + B_\theta \cos\nu\theta \tag{3}$$

Constraint 5 shows that ν is a non-negative integer.

$$R'' + \frac{R'}{r} + \left(k^2 - \frac{\nu^2}{r^2}\right) R = 0 \tag{4}$$

Here the primes denote derivative with respect to r. Now, we'll follow Jackson and make the substitution $x = kr$, $dx = kdr$

$$\ddot{R} + \frac{\dot{R}}{x} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0 \Rightarrow R(x) = A_r J_\nu(x) + B_r N_\nu(x) \tag{5}$$

Constraint 3 shows that $B_r = 0$, and by choosing the Bessel functions in the first place, constraint 4 is already satisfied.

So putting the whole solution back together, and considering all possibilities for n and ν.

$$\Phi = \sum_{\nu=0}^{\infty} \int_0^\infty J_\nu(kr)\sinh(kz)\left[A_\nu(k)\sin(\nu\theta) + B_\nu(k)\cos(\nu\theta)\right] \mathrm{d}k \tag{6}$$

We are now left to calculate the coefficients $A_\nu(k)$ and $B_\nu(k)$. Applying the boundary condition on V(r) on the top face, we get,

$$V(r) = \sum_\nu \int J_\nu(kr)\sinh(kL)\left[A_\nu(k)\sin(\nu\theta) + B_\nu(k)\cos(\nu\theta)\right] \mathrm{d}k \tag{7}$$

The path naturally diverges here as the $A_\nu(k)$'s must be calculated separetely from the $B_\nu(k)$'s. I'll calculate the $B_\nu(k)$'s here, and nothe that the $A_\nu(k)$'s can be calculted in a nearly identical way. Multiplying both sides of the previous equation by $\cos(\nu'\theta)$ and integrating gives,

$$V(r)\int_0^{2\pi}\cos(\nu'\theta)\mathrm{d}\theta = \sum_\nu \int_0^\infty J_\nu(kr)\sinh(kL)\int_0^{2\pi}\left[A_\nu(k)\sin(\nu\theta)\cos(\nu'\theta) + B_\nu(k)\cos(\nu\theta)\cos(\nu'\theta)\right]\mathrm{d}\theta\mathrm{d}k \tag{8}$$

By orthogonality of the sines and cosines, the $A_\nu(k)$ term is zero, and the $B_\nu(k)$ term colapses according to $\int_0^{2\pi}\cos(\nu\theta)\cos(\nu'\theta)\mathrm{d}\theta = \pi\delta_{\nu\nu'}\forall\nu \neq 0$. When $\nu = 0$ the coeficient becomes 2π instead of just π. To keep the functional form, I'll use the more common case, and keep that factor of two in mind for when we need it.

$$V(r)\int_0^{2\pi}\cos(\nu'\theta)\mathrm{d}\theta = \pi\int_0^\infty J_{\nu'}(kr)\sinh(kL)B_{\nu'}(k)\mathrm{d}k \tag{9}$$

That defines the $B_\nu(k)$'s, but it remains to solve explicitly for them. The trick here is to multiply both sides by $rJ_{\nu'}(k'r)$ and integrate over r.

$$\frac{1}{\pi}\int_0^{2\pi}\cos(\nu'\theta)\mathrm{d}\theta\int_0^\infty rV(r)J_{\nu'}(k'r)\mathrm{d}r = \int_0^\infty B_{\nu'}(k)\sinh(kL)\int_0^\infty rJ_{\nu'}(k'r)J_{\nu'}(kr)\mathrm{d}r\mathrm{d}k \tag{10}$$

The integral over r is the Hankel transformation so we can use the identity that Jackson lists as 3.108, $\int_0^\infty rJ_{\nu'}(k'r)J_{\nu'}(kr)\mathrm{d}r = \frac{1}{k}\delta(k'-k)$.

$$B_{\nu'}(k') = \frac{k'}{\pi\sinh(k'L)}\int_0^{2\pi}\cos(\nu'\theta)\mathrm{d}\theta\int_0^\infty rV(r)J_{\nu'}(k'r)\mathrm{d}r \tag{11}$$

The theta integral in the preceeding equation goes to zero for all ν' except zero. Since we're left only with $\nu' = 0$ let's put that factor of two back in, and at the same time put in our particular form for V(r).

$$B_0(k) = \frac{Vk}{\sinh(kL)}\int_0^a rJ_\nu(kr)\mathrm{d}r = \frac{Va}{\sinh(kL)}J_1(ka) \tag{12}$$

So finally, we can put it back in to Φ.

$$\boxed{\Phi = Va\int_0^\infty J_0(kr)J_1(ka)\frac{\sinh(kz)}{\sinh(kL)}\mathrm{d}k} \tag{13}$$

If we want to make this look like it does in Jackson (which I don't really) we can make the substitution $\lambda = ka$.

$$\Phi = V\int_0^\infty J_0(\lambda\frac{r}{a})J_1(\lambda)\frac{\sinh(\lambda\frac{z}{a})}{\sinh(\lambda\frac{L}{a})}\mathrm{d}\lambda \tag{14}$$